

## 16.5 Second-Order Conservative Equations

Usually when you have a system of high-order differential equations to solve it is best to reformulate them as a system of first-order equations, as discussed in §16.0. There is a particular class of equations that occurs quite frequently in practice where you can gain about a factor of two in efficiency by differencing the equations directly. The equations are second-order systems where the derivative does not appear on the right-hand side:

$$y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = z_0 \quad (16.5.1)$$

As usual,  $y$  can denote a vector of values.

*Stoermer's rule*, dating back to 1907, has been a popular method for discretizing such systems. With  $h = H/m$  we have

$$\begin{aligned} y_1 &= y_0 + h[z_0 + \frac{1}{2}hf(x_0, y_0)] \\ y_{k+1} - 2y_k + y_{k-1} &= h^2 f(x_0 + kh, y_k), \quad k = 1, \dots, m-1 \\ z_m &= (y_m - y_{m-1})/h + \frac{1}{2}hf(x_0 + H, y_m) \end{aligned} \quad (16.5.2)$$

Here  $z_m$  is  $y'(x_0 + H)$ . Henrici showed how to rewrite equations (16.5.2) to reduce roundoff error by using the quantities  $\Delta_k \equiv y_{k+1} - y_k$ . Start with

$$\begin{aligned} \Delta_0 &= h[z_0 + \frac{1}{2}hf(x_0, y_0)] \\ y_1 &= y_0 + \Delta_0 \end{aligned} \quad (16.5.3)$$

Then for  $k = 1, \dots, m-1$ , set

$$\begin{aligned} \Delta_k &= \Delta_{k-1} + h^2 f(x_0 + kh, y_k) \\ y_{k+1} &= y_k + \Delta_k \end{aligned} \quad (16.5.4)$$

Finally compute the derivative from

$$z_m = \Delta_{m-1}/h + \frac{1}{2}hf(x_0 + H, y_m) \quad (16.5.5)$$

Gragg again showed that the error series for equations (16.5.3)–(16.5.5) contains only even powers of  $h$ , and so the method is a logical candidate for extrapolation à la Bulirsch-Stoer. We replace `mmid` by the following routine `stoerm`:

```

SUBROUTINE stoerm(y,d2y,nv,xs,htot,nstep,yout,derivs)
  INTEGER nstep,nv,NMAX
  REAL htot,xs,d2y(nv),y(nv),yout(nv)
  EXTERNAL derivs
  PARAMETER (NMAX=50)           Maximum value of nv.
C  USES derivs
  Stoermer's rule for integrating  $y'' = f(x, y)$  for a system of  $n = nv/2$  equations. On input
  y(1:nv) contains  $y$  in its first  $n$  elements and  $y'$  in its second  $n$  elements, all evaluated
  at xs. d2y(1:nv) contains the right-hand side function  $f$  (also evaluated at xs) in its
  first  $n$  elements. Its second  $n$  elements are not referenced. Also input is htot, the total
  step to be taken, and nstep, the number of substeps to be used. The output is returned
  as yout(1:nv), with the same storage arrangement as y. derivs is the user-supplied
  subroutine that calculates  $f$ .
  INTEGER i,n,neqns,nn
  REAL h,h2,halfh,x,ytemp(NMAX)
  h=htot/nstep           Stepsize this trip.
  halfh=0.5*h
  neqns=nv/2           Number of equations.
  do 11 i=1,neqns       First step.
    n=neqns+i
    ytemp(n)=h*(y(n)+halfh*d2y(i))
  
```

```

      ytemp(i)=y(i)+ytemp(n)
enddo 11
x=xs+h
call derivs(x,ytemp,yout)      Use yout for temporary storage of derivatives.
h2=h*h
do 13 nn=2,nstep              General step.
  do 12 i=1,neqns
    n=neqns+i
    ytemp(n)=ytemp(n)+h2*yout(i)
    ytemp(i)=ytemp(i)+ytemp(n)
  enddo 12
  x=x+h
  call derivs(x,ytemp,yout)
enddo 13
do 14 i=1,neqns              Last step.
  n=neqns+i
  yout(n)=ytemp(n)/h+halfh*yout(i)
  yout(i)=ytemp(i)
enddo 14
return
END

```

Note that for compatibility with `bsstep` the arrays `y` and `d2y` are of length  $2n$  for a system of  $n$  second-order equations. The values of  $y$  are stored in the first  $n$  elements of `y`, while the first derivatives are stored in the second  $n$  elements. The right-hand side  $f$  is stored in the first  $n$  elements of the array `d2y`; the second  $n$  elements are unused. With this storage arrangement you can use `bsstep` simply by replacing the call to `mmid` with one to `stoerm` using the same arguments; just be sure that the argument `nv` of `bsstep` is set to  $2n$ . You should also use the more efficient sequence of stepsizes suggested by Deuffhard:

$$n = 1, 2, 3, 4, 5, \dots \quad (16.5.6)$$

and set `KMAXX = 12` in `bsstep`.

#### CITED REFERENCES AND FURTHER READING:

Deuffhard, P. 1985, *SIAM Review*, vol. 27, pp. 505–535.

## 16.6 Stiff Sets of Equations

As soon as one deals with more than one first-order differential equation, the possibility of a *stiff* set of equations arises. Stiffness occurs in a problem where there are two or more very different scales of the independent variable on which the dependent variables are changing. For example, consider the following set of equations [1]:

$$\begin{aligned} u' &= 998u + 1998v \\ v' &= -999u - 1999v \end{aligned} \quad (16.6.1)$$

with boundary conditions

$$u(0) = 1 \quad v(0) = 0 \quad (16.6.2)$$