

Chi-Square Probability Function

$P(\chi^2|\nu)$ is defined as the probability that the observed chi-square for a correct model should be less than a value χ^2 . (We will discuss the use of this function in Chapter 15.) Its complement $Q(\chi^2|\nu)$ is the probability that the observed chi-square will exceed the value χ^2 by chance *even* for a correct model. In both cases ν is an integer, the number of degrees of freedom. The functions have the limiting values

$$P(0|\nu) = 0 \quad P(\infty|\nu) = 1 \quad (6.2.16)$$

$$Q(0|\nu) = 1 \quad Q(\infty|\nu) = 0 \quad (6.2.17)$$

and the following relation to the incomplete gamma functions,

$$P(\chi^2|\nu) = P\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) = \text{gammp}\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) \quad (6.2.18)$$

$$Q(\chi^2|\nu) = Q\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) = \text{gammq}\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) \quad (6.2.19)$$

CITED REFERENCES AND FURTHER READING:

Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, Volume 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapters 6, 7, and 26.

Pearson, K. (ed.) 1951, *Tables of the Incomplete Gamma Function* (Cambridge: Cambridge University Press).

6.3 Exponential Integrals

The standard definition of the exponential integral is

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt, \quad x > 0, \quad n = 0, 1, \dots \quad (6.3.1)$$

The function defined by the principal value of the integral

$$\text{Ei}(x) = - \int_{-x}^\infty \frac{e^{-t}}{t} dt = \int_{-\infty}^x \frac{e^t}{t} dt, \quad x > 0 \quad (6.3.2)$$

is also called an exponential integral. Note that $\text{Ei}(-x)$ is related to $-E_1(x)$ by analytic continuation.

The function $E_n(x)$ is a special case of the incomplete gamma function

$$E_n(x) = x^{n-1} \Gamma(1-n, x) \quad (6.3.3)$$

We can therefore use a similar strategy for evaluating it. The continued fraction — just equation (6.2.6) rewritten — converges for all $x > 0$:

$$E_n(x) = e^{-x} \left(\frac{1}{x+} \frac{n}{1+} \frac{1}{x+} \frac{n+1}{1+} \frac{2}{x+} \dots \right) \quad (6.3.4)$$

We use it in its more rapidly converging even form,

$$E_n(x) = e^{-x} \left(\frac{1}{x+n-} \frac{1 \cdot n}{x+n+2-} \frac{2(n+1)}{x+n+4-} \dots \right) \quad (6.3.5)$$

The continued fraction only really converges fast enough to be useful for $x \gtrsim 1$. For $0 < x \lesssim 1$, we can use the series representation

$$E_n(x) = \frac{(-x)^{n-1}}{(n-1)!} [-\ln x + \psi(n)] - \sum_{\substack{m=0 \\ m \neq n-1}}^{\infty} \frac{(-x)^m}{(m-n+1)m!} \quad (6.3.6)$$

The quantity $\psi(n)$ here is the digamma function, given for integer arguments by

$$\psi(1) = -\gamma, \quad \psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \quad (6.3.7)$$

where $\gamma = 0.5772156649\dots$ is Euler's constant. We evaluate the expression (6.3.6) in order of ascending powers of x :

$$E_n(x) = - \left[\frac{1}{(1-n)} - \frac{x}{(2-n) \cdot 1} + \frac{x^2}{(3-n)(1 \cdot 2)} - \dots + \frac{(-x)^{n-2}}{(-1)(n-2)!} \right] \\ + \frac{(-x)^{n-1}}{(n-1)!} [-\ln x + \psi(n)] - \left[\frac{(-x)^n}{1 \cdot n!} + \frac{(-x)^{n+1}}{2 \cdot (n+1)!} + \dots \right] \quad (6.3.8)$$

The first square bracket is omitted when $n = 1$. This method of evaluation has the advantage that for large n the series converges before reaching the term containing $\psi(n)$. Accordingly, one needs an algorithm for evaluating $\psi(n)$ only for small n , $n \lesssim 20-40$. We use equation (6.3.7), although a table look-up would improve efficiency slightly.

Amos [1] presents a careful discussion of the truncation error in evaluating equation (6.3.8), and gives a fairly elaborate termination criterion. We have found that simply stopping when the last term added is smaller than the required tolerance works about as well.

Two special cases have to be handled separately:

$$E_0(x) = \frac{e^{-x}}{x} \\ E_n(0) = \frac{1}{n-1}, \quad n > 1 \quad (6.3.9)$$

The routine `expint` allows fast evaluation of $E_n(x)$ to any accuracy `EPS` within the reach of your machine's word length for floating-point numbers. The only modification required for increased accuracy is to supply Euler's constant with enough significant digits. Wrench [2] can provide you with the first 328 digits if necessary!

```

FUNCTION expint(n,x)
INTEGER n,MAXIT
REAL expint,x,EPS,FPMIN,EULER
PARAMETER (MAXIT=100,EPS=1.e-7,FPMIN=1.e-30,EULER=.5772156649)
    Evaluates the exponential integral  $E_n(x)$ .
    Parameters: MAXIT is the maximum allowed number of iterations; EPS is the desired relative error, not smaller than the machine precision; FPMIN is a number near the smallest representable floating-point number; EULER is Euler's constant  $\gamma$ .
INTEGER i,ii,nm1
REAL a,b,c,d,del,fact,h,psi
nm1=n-1
if(n.lt.0.or.x.lt.0..or.(x.eq.0..and.(n.eq.0.or.n.eq.1)))then
    pause 'bad arguments in expint'
else if(n.eq.0)then
    expint=exp(-x)/x
    Special case.
else if(x.eq.0.)then
    expint=1./nm1
    Another special case.
else if(x.gt.1.)then
    Lenz's algorithm (§5.2).
    b=x+n
    c=1./FPMIN
    d=1./b
    h=d
    do 11 i=1,MAXIT
        a=-i*(nm1+i)
        b=b+2.
        d=1./(a*d+b)
        Denominators cannot be zero.
        c=b+a/c
        del=c*d
        h=h*del
        if(abs(del-1.)<.EPS)then
            expint=h*exp(-x)
            return
        endif
    enddo 11
    pause 'continued fraction failed in expint'
else
    Evaluate series.
    if(nm1.ne.0)then
        Set first term.
        expint=1./nm1
    else
        expint=-log(x)-EULER
    endif
    fact=1.
    do 13 i=1,MAXIT
        fact=-fact*x/i
        if(i.ne.nm1)then
            del=-fact/(i-nm1)
        else
            psi=-EULER
            Compute  $\psi(n)$ .
            do 12 ii=1,nm1
                psi=psi+1./ii
            enddo 12
            del=fact*(-log(x)+psi)
        endif
        expint=expint+del
        if(abs(del)<.EPS) return
    enddo 13

```

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```

    pause 'series failed in expint'
endif
return
END

```

A good algorithm for evaluating Ei is to use the power series for small x and the asymptotic series for large x . The power series is

$$Ei(x) = \gamma + \ln x + \frac{x}{1 \cdot 1!} + \frac{x^2}{2 \cdot 2!} + \cdots \quad (6.3.10)$$

where γ is Euler's constant. The asymptotic expansion is

$$Ei(x) \sim \frac{e^x}{x} \left(1 + \frac{1!}{x} + \frac{2!}{x^2} + \cdots \right) \quad (6.3.11)$$

The lower limit for the use of the asymptotic expansion is approximately $|\ln EPS|$, where EPS is the required relative error.

```

FUNCTION ei(x)
INTEGER MAXIT
REAL ei,x,EPS,EULER,FPMIN
PARAMETER (EPS=6.e-8,EULER=.57721566,MAXIT=100,FPMIN=1.e-30)
    Computes the exponential integral  $Ei(x)$  for  $x > 0$ .
    Parameters:  $EPS$  is the relative error, or absolute error near the zero of  $Ei$  at  $x = 0.3725$ ;
     $EULER$  is Euler's constant  $\gamma$ ;  $MAXIT$  is the maximum number of iterations allowed;  $FPMIN$ 
    is a number near the smallest representable floating-point number.
INTEGER k
REAL fact,prev,sum,term
if(x.le.0.) pause 'bad argument in ei'
if(x.lt.FPMIN)then
    ei=log(x)+EULER
    Special case: avoid failure of convergence test because of underflow.
else if(x.le.-log(EPS))then
    Use power series.
    sum=0.
    fact=1.
    do 11 k=1,MAXIT
        fact=fact*x/k
        term=fact/k
        sum=sum+term
        if(term.lt.EPS*sum)goto 1
    enddo 11
    pause 'series failed in ei'
1    ei=sum+log(x)+EULER
else
    Use asymptotic series.
    Start with second term.
    sum=0.
    term=1.
    do 12 k=1,MAXIT
        prev=term
        term=term*k/x
        if(term.lt.EPS)goto 2
        if(term.lt.prev)then
            sum=sum+term
            Still converging: add new term.
        else
            sum=sum-prev
            Diverging: subtract previous term and exit.
            goto 2
        endif
    enddo 12
2    ei=exp(x)*(1.+sum)/x
endif

```

```
return
END
```

CITED REFERENCES AND FURTHER READING:

Stegun, I.A., and Zucker, R. 1974, *Journal of Research of the National Bureau of Standards*, vol. 78B, pp. 199–216; 1976, *op. cit.*, vol. 80B, pp. 291–311.
 Amos D.E. 1980, *ACM Transactions on Mathematical Software*, vol. 6, pp. 365–377 [1]; also vol. 6, pp. 420–428.
 Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, Volume 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapter 5.
 Wrench J.W. 1952, *Mathematical Tables and Other Aids to Computation*, vol. 6, p. 255. [2]

6.4 Incomplete Beta Function, Student's Distribution, F-Distribution, Cumulative Binomial Distribution

The incomplete beta function is defined by

$$I_x(a, b) \equiv \frac{B_x(a, b)}{B(a, b)} \equiv \frac{1}{B(a, b)} \int_0^x t^{a-1}(1-t)^{b-1} dt \quad (a, b > 0) \quad (6.4.1)$$

It has the limiting values

$$I_0(a, b) = 0 \quad I_1(a, b) = 1 \quad (6.4.2)$$

and the symmetry relation

$$I_x(a, b) = 1 - I_{1-x}(b, a) \quad (6.4.3)$$

If a and b are both rather greater than one, then $I_x(a, b)$ rises from “near-zero” to “near-unity” quite sharply at about $x = a/(a + b)$. Figure 6.4.1 plots the function for several pairs (a, b) .

The incomplete beta function has a series expansion

$$I_x(a, b) = \frac{x^a(1-x)^b}{aB(a, b)} \left[1 + \sum_{n=0}^{\infty} \frac{B(a+1, n+1)}{B(a+b, n+1)} x^{n+1} \right], \quad (6.4.4)$$

but this does not prove to be very useful in its numerical evaluation. (Note, however, that the beta functions in the coefficients can be evaluated for each value of n with just the previous value and a few multiplies, using equations 6.1.9 and 6.1.3.)

The continued fraction representation proves to be much more useful,

$$I_x(a, b) = \frac{x^a(1-x)^b}{aB(a, b)} \left[\frac{1}{1+} \frac{d_1}{1+} \frac{d_2}{1+} \dots \right] \quad (6.4.5)$$

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