

```

      p11=(x*(2*11-1)*pmmp1-(11+m-1)*pmm)/(11-m)
      pmm=pmmp1
      pmmp1=p11
    enddo 12
    plgndr=p11
  endif
endif
return
END

```

## CITED REFERENCES AND FURTHER READING:

- Magnus, W., and Oberhettinger, F. 1949, *Formulas and Theorems for the Functions of Mathematical Physics* (New York: Chelsea), pp. 54ff. [1]
- Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, Volume 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapter 8. [2]

## 6.9 Fresnel Integrals, Cosine and Sine Integrals

### Fresnel Integrals

The two Fresnel integrals are defined by

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt, \quad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt \quad (6.9.1)$$

The most convenient way of evaluating these functions to arbitrary precision is to use power series for small  $x$  and a continued fraction for large  $x$ . The series are

$$\begin{aligned}
 C(x) &= x - \left(\frac{\pi}{2}\right)^2 \frac{x^5}{5 \cdot 2!} + \left(\frac{\pi}{2}\right)^4 \frac{x^9}{9 \cdot 4!} - \dots \\
 S(x) &= \left(\frac{\pi}{2}\right) \frac{x^3}{3 \cdot 1!} - \left(\frac{\pi}{2}\right)^3 \frac{x^7}{7 \cdot 3!} + \left(\frac{\pi}{2}\right)^5 \frac{x^{11}}{11 \cdot 5!} - \dots
 \end{aligned} \quad (6.9.2)$$

There is a complex continued fraction that yields both  $S(x)$  and  $C(x)$  simultaneously:

$$C(x) + iS(x) = \frac{1+i}{2} \operatorname{erf} z, \quad z = \frac{\sqrt{\pi}}{2}(1-i)x \quad (6.9.3)$$

where

$$\begin{aligned}
 e^{z^2} \operatorname{erfc} z &= \frac{1}{\sqrt{\pi}} \left( \frac{1}{z + \frac{1/2}{z + \frac{1}{z + \frac{3/2}{z + \frac{2}{\dots}}}}} \right) \\
 &= \frac{2z}{\sqrt{\pi}} \left( \frac{1}{2z^2 + 1 - \frac{1 \cdot 2}{2z^2 + 5 - \frac{3 \cdot 4}{2z^2 + 9 - \dots}}} \right)
 \end{aligned} \quad (6.9.4)$$

In the last line we have converted the “standard” form of the continued fraction to its “even” form (see §5.2), which converges twice as fast. We must be careful not to evaluate the alternating series (6.9.2) at too large a value of  $x$ ; inspection of the terms shows that  $x = 1.5$  is a good point to switch over to the continued fraction.

Note that for large  $x$

$$C(x) \sim \frac{1}{2} + \frac{1}{\pi x} \sin\left(\frac{\pi}{2}x^2\right), \quad S(x) \sim \frac{1}{2} - \frac{1}{\pi x} \cos\left(\frac{\pi}{2}x^2\right) \quad (6.9.5)$$

Thus the precision of the routine `frenel` may be limited by the precision of the library routines for sine and cosine for large  $x$ .

```

SUBROUTINE frenel(x,s,c)
  INTEGER MAXIT
  REAL c,s,x,EPS,FPMIN,PI,PIBY2,XMIN
  PARAMETER (EPS=6.e-8,MAXIT=100,FPMIN=1.e-30,XMIN=1.5,
  *      PI=3.1415927,PIBY2=1.5707963)
  Computes the Fresnel integrals  $S(x)$  and  $C(x)$  for all real  $x$ .
  Parameters: EPS is the relative error; MAXIT is the maximum number of iterations allowed;
  FPMIN is a number near the smallest representable floating-point number; XMIN is the
  dividing line between using the series and continued fraction;  $PI = \pi$ ;  $PIBY2 = \pi/2$ .
  INTEGER k,n
  REAL a,absc,ax,fact,pix2,sign,sum,sumc,sums,term,test
  COMPLEX b,cc,d,h,del,cs
  LOGICAL odd
  absc(h)=abs(real(h))+abs(aimag(h))   Statement function.
  ax=abs(x)
  if(ax.lt.sqrt(FPMIN))then           Special case: avoid failure of convergence test
    s=0.                               because of underflow.
    c=ax
  else if(ax.le.XMIN)then            Evaluate both series simultaneously.
    sum=0.
    sums=0.
    sumc=ax
    sign=1.
    fact=PIBY2*ax*ax
    odd=.true.
    term=ax
    n=3
    do 11 k=1,MAXIT
      term=term*fact/k
      sum=sum+sign*term/n
      test=abs(sum)*EPS
      if(odd)then
        sign=-sign
        sums=sum
        sum=sumc
      else
        sumc=sum
        sum=sums
      endif
      if(term.lt.test)goto 1
      odd=.not.odd
      n=n+2
    enddo 11
    pause 'series failed in frenel'
  1  s=sums
    c=sumc
  else                                Evaluate continued fraction by modified Lentz's
    pix2=PI*ax*ax                      method (§5.2).
    b=cplx(1.,-pix2)

```

```

cc=1./FPMIN
d=1./b
h=d
n=-1
do 12 k=2,MAXIT
  n=n+2
  a=-n*(n+1)
  b=b+4.
  d=1./(a*d+b)          Denominators cannot be zero.
  cc=b+a/cc
  del=cc*d
  h=h*del
  if(absc(del-1.).lt.EPS)goto 2
enddo 12
pause 'cf failed in frenel'
2 h=h*cmlpx(ax,-ax)
cs=cmlpx(.5,.5)*(1.-cmlpx(cos(.5*pix2),sin(.5*pix2))*h)
c=real(cs)
s=aimag(cs)
endif
if(x.lt.0.)then          Use antisymmetry.
  c=-c
  s=-s
endif
return
END

```

## Cosine and Sine Integrals

The cosine and sine integrals are defined by

$$\begin{aligned} \text{Ci}(x) &= \gamma + \ln x + \int_0^x \frac{\cos t - 1}{t} dt \\ \text{Si}(x) &= \int_0^x \frac{\sin t}{t} dt \end{aligned} \quad (6.9.6)$$

Here  $\gamma \approx 0.5772\dots$  is Euler's constant. We only need a way to calculate the functions for  $x > 0$ , because

$$\text{Si}(-x) = -\text{Si}(x), \quad \text{Ci}(-x) = \text{Ci}(x) - i\pi \quad (6.9.7)$$

Once again we can evaluate these functions by a judicious combination of power series and complex continued fraction. The series are

$$\begin{aligned} \text{Si}(x) &= x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \dots \\ \text{Ci}(x) &= \gamma + \ln x + \left( -\frac{x^2}{2 \cdot 2!} + \frac{x^4}{4 \cdot 4!} - \dots \right) \end{aligned} \quad (6.9.8)$$

The continued fraction for the exponential integral  $E_1(ix)$  is

$$\begin{aligned} E_1(ix) &= -\text{Ci}(x) + i[\text{Si}(x) - \pi/2] \\ &= e^{-ix} \left( \frac{1}{ix + 1} - \frac{1}{1 + ix} + \frac{1}{ix + 1} - \frac{2}{1 + ix} + \frac{2}{ix + 1} - \dots \right) \\ &= e^{-ix} \left( \frac{1}{1 + ix} - \frac{1^2}{3 + ix} + \frac{2^2}{5 + ix} - \dots \right) \end{aligned} \quad (6.9.9)$$

The “even” form of the continued fraction is given in the last line and converges twice as fast for about the same amount of computation. A good crossover point from the alternating series to the continued fraction is  $x = 2$  in this case. As for the Fresnel integrals, for large  $x$  the precision may be limited by the precision of the sine and cosine routines.

```

SUBROUTINE cisi(x,ci,si)
INTEGER MAXIT
REAL ci,si,x,EPS,EULER,PIBY2,FPMIN,TMIN
PARAMETER (EPS=6.e-8,EULER=.57721566,MAXIT=100,PIBY2=1.5707963,
* FPMIN=1.e-30,TMIN=2.)
  Computes the cosine and sine integrals  $Ci(x)$  and  $Si(x)$ .  $Ci(0)$  is returned as a large negative
  number and no error message is generated. For  $x < 0$  the routine returns  $Ci(-x)$  and you
  must supply the  $-i\pi$  yourself.
  Parameters: EPS is the relative error, or absolute error near a zero of  $Ci(x)$ ; EULER =  $\gamma$ ;
  MAXIT is the maximum number of iterations allowed; PIBY2 =  $\pi/2$ ; FPMIN is a number
  near the smallest representable floating-point number; TMIN is the dividing line between
  using the series and continued fraction.
INTEGER i,k
REAL a,err,fact,sign,sum,sumc,sums,t,term,absc
COMPLEX h,b,c,d,del
LOGICAL odd
absc(h)=abs(real(h))+abs(aimag(h))      Statement function.
t=abs(x)
if(t.eq.0.)then                        Special case.
  si=0.
  ci=-1./FPMIN
  return
endif
if(t.gt.TMIN)then                      Evaluate continued fraction by modified Lentz's
  b=cplx(1.,t)                          method (§5.2).
  c=1./FPMIN
  d=1./b
  h=d
  do 11 i=2,MAXIT
    a=-(i-1)**2
    b=b+2.
    d=1./(a*d+b)                        Denominators cannot be zero.
    c=b+a/c
    del=c*d
    h=h*del
    if(absc(del-1.).lt.EPS)goto 1
  enddo 11
  pause 'cf failed in cisi'
  continue
  h=cplx(cos(t),-sin(t))*h
  ci=-real(h)
  si=PIBY2+aimag(h)
else
  Evaluate both series simultaneously.
  Special case: avoid failure of convergence test
  because of underflow.
  if(t.lt.sqrt(FPMIN))then
    sumc=0.
    sums=t
  else
    sum=0.
    sums=0.
    sumc=0.
    sign=1.
    fact=1.
    odd=.true.
    do 12 k=1,MAXIT
      fact=fact*t/k
      term=fact/k

```

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```

        sum=sum+sign*term
        err=term/abs(sum)
        if(odd)then
            sign=-sign
            sums=sum
            sum=sumc
        else
            sumc=sum
            sum=sums
        endif
        if(err.lt.EPS)goto 2
        odd=.not.odd
    enddo i2
    pause 'maxits exceeded in cisi'
endif
si=sums
ci=sumc+log(t)+EULER
endif
if(x.lt.0.)si=-si
return
END

```

## CITED REFERENCES AND FURTHER READING:

- Stegun, I.A., and Zucker, R. 1976, *Journal of Research of the National Bureau of Standards*, vol. 80B, pp. 291–311; 1981, *op. cit.*, vol. 86, pp. 661–686.
- Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, Volume 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapters 5 and 7.

## 6.10 Dawson's Integral

Dawson's Integral  $F(x)$  is defined by

$$F(x) = e^{-x^2} \int_0^x e^{t^2} dt \quad (6.10.1)$$

The function can also be related to the complex error function by

$$F(z) = \frac{i\sqrt{\pi}}{2} e^{-z^2} [1 - \operatorname{erfc}(-iz)]. \quad (6.10.2)$$

A remarkable approximation for  $F(x)$ , due to Rybicki [1], is

$$F(z) = \lim_{h \rightarrow 0} \frac{1}{\sqrt{\pi}} \sum_{n \text{ odd}} \frac{e^{-(z-nh)^2}}{n} \quad (6.10.3)$$

What makes equation (6.10.3) unusual is that its accuracy increases *exponentially* as  $h$  gets small, so that quite moderate values of  $h$  (and correspondingly quite rapid convergence of the series) give very accurate approximations.